

Universal Entropy Bound for Rotating Systems

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(February 5, 2008)

Abstract

We conjecture a *universal upper bound* to the entropy of a *rotating* system. The entropy bound follows from application of the generalized second law of thermodynamics to an idealized gedanken experiment in which an entropy-bearing rotating system falls into a black hole. This bound is *stronger* than the Bekenstein entropy bound for non-rotating systems.

One of the most intriguing features of both the classical and quantum theory of black-holes is the striking analogy between the laws of black-hole physics and the universal laws of thermodynamics [1–6]. In particular, Hawking’s (classical) theorem [3], “The surface area of a black hole never decreases,” is a property reminiscent of the entropy of a closed system. This striking analogy had led Bekenstein [7–9] to conjecture that the area of a black hole (in suitable units) may be regarded as the black-hole entropy – entropy in the sense of information about the black-hole interior inaccessible to observers outside the black hole. This conjecture is logically related to a second conjecture, known as *the generalized second law of thermodynamics* (GSL): “*The sum of the black-hole entropy* (now known to be $\frac{1}{4}$ of the horizon’s surface area) *and the common (ordinary) entropy in the black-hole exterior never decreases*”.

The general belief in the validity of the ordinary second law of thermodynamics rests mainly on the repeated failure over the years of attempts to violate it. There currently exists no general proof of the law based on the known microscopic laws of physics. In the

analog case of the GSL considerably less is known since the fundamental microscopic laws of physics, namely, the laws of quantum gravity are not yet known. Hence, one is forced to consider gedanken experiments in order to test the validity of the GSL. Such experiments are important since the validity of the GSL underlies the relationship between black-hole physics and thermodynamics. If the GSL is valid, then it is very plausible that the laws of black-hole physics are simply the ordinary laws of thermodynamics applied to a self-gravitating quantum system. This conclusion, if true, would provide a striking demonstration of the *unity* of physics. Thus, it is of considerable interest to test the validity of the GSL in various gedanken experiments.

In a *classical* context, a basic physical mechanism is known by which a violation of the GSL can be achieved: Consider a box filled with matter of proper energy E and entropy S which is dropped into a black hole. The energy delivered to the black hole can be arbitrarily *red-shifted* by letting the assimilation point approach the black-hole horizon. As shown by Bekenstein [9,10], if the box is deposited with no radial momentum a proper distance R above the horizon, and then allowed to fall in such that

$$R < \hbar S / 2\pi E , \quad (1)$$

then the black-hole area increase (or equivalently, the increase in black-hole entropy) is not large enough to compensate for the decrease of S in common (ordinary) entropy. Bekenstein has proposed a resolution of this apparent violation of the GSL which is based on the *quantum* nature of the matter dropped into the black hole. He has proposed the existence of a universal upper bound on the entropy S of any system of total energy E and maximal radius R [11]:

$$S \leq 2\pi R E / \hbar . \quad (2)$$

It has been argued [11–13], and disputed [14,15] that this restriction is *necessary* for enforcement of the GSL; the box's entropy disappears but an increase in black-hole entropy occurs which ensures that the GSL is respected provided S is bounded as in Eq. (2). Other derivations of the universal bound Eq. (2) which are based on black-hole physics have been given

by Zaslavskii [16–18] and by Li and Liu [19]. Few pieces of evidence exist concerning the validity of the bound for self-gravitating systems [16,20,21]. However, the universal bound Eq. (2) is known to be true independently of black-hole physics for a variety of systems in which gravity is negligible [22–25]. In particular, Schiffer and Bekenstein [24] had provided an analytic proof of the bound for free scalar, electromagnetic and massless spinor fields enclosed in boxes of arbitrary shape and topology.

In this paper we test the validity of the GSL in an idealized gedanken experiment in which an entropy-bearing *rotating* system falls into a stationary black hole. We argue that while the bound Eq. (2) may be a necessary condition for the fulfillment of the GSL, it may *not* be a sufficient one.

It is not difficult to see why a *stronger* upper bound must exist for the entropy S of an arbitrary system with energy E , intrinsic angular momentum s and (maximal) radius R : The *gravitational spin-orbit interaction* [26] (the analog of the more familiar electromagnetic spin-orbit interaction) experienced by the spinning body (which, of course, was not relevant in the above mentioned gedanken experiment) can decrease the energy delivered to the black hole. This would *decrease* the change in black-hole entropy (area). Hence, the GSL will be violated unless the spinning-system entropy (what disappears from the black-hole exterior) is restricted by a bound *stronger* than Eq. (2).

Furthermore, there is one disturbing feature of the universal bound Eq. (2). As was pointed out by Bekenstein [11], Kerr black holes conform to the bound; however, only the Schwarzschild hole actually *attains* the bound. This uniqueness of the Schwarzschild black hole (in the sense that it is the *only* black hole which have the maximum entropy allowed by quantum theory and general relativity) among the electrically neutral Kerr-family solutions is somewhat disturbing. Clearly, the unity of physics demands a stronger bound for *rotating* systems in general, and for black holes in particular (see also [27]).

In fact, the plausible existence of an upper bound stronger than Eq. (2) on the entropy of a rotating system has nothing to do with black-hole physics. Classically, entropy is a measure of the phase space available to the system in question. Consider a system whose energy is

no more than E . The limitation imposed on E amounts to a limitation on the momentum space available to the system's components (provided the potential energy is bounded from below). Now, if part of the system's energy is in the form of a coherent (global) kinetic energy (in contrast to random motion of its constituents), then the momentum space available to the system's components is *further* limited (part of the energy of the system is irrelevant for the system's statistical properties). If the system has a finite dimension in space, then its phase space is limited. This amounts to an upper bound on its entropy. This bound evidently *decreases* with the absolute value of the intrinsic angular momentum of the system. However, our simple argument cannot yield the exact dependence of the entropy bound on the system's parameters: its energy, intrinsic angular momentum (spin), and proper radius.

In fact, black-hole physics (more precisely, the GSL) provides a concrete universal upper bound for rotating systems. We consider a spinning body of rest mass μ , (intrinsic) spin s and proper cylindrical radius R , which is descending into a black hole. We consider plane (equatorial) motions of the body in a Kerr-Newman background [28], with the (intrinsic) spin orthogonal to the plane (the general motion of a spinning particle in a Kerr-Newman background is very complicated, and has not been analyzed so far). The black-hole (event and inner) horizons are located at

$$r_{\pm} = M \pm (M^2 - Q^2 - a^2)^{1/2} , \quad (3)$$

where M , Q and a are the mass, charge and angular-momentum per unit mass of the hole, respectively (we use gravitational units in which $G = c = 1$). The test particle approximation implies $|s|/(\mu r_+) \ll 1$.

The equation of motion of a spinning body in the equatorial plane of a Kerr-Newman background is a quadratic equation for the conserved energy (energy-at-infinity) E of the body [29]

$$\tilde{\alpha}E^2 - 2\tilde{\beta}E + \tilde{\gamma} = 0 , \quad (4)$$

where the expression for $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are given in [29].

The actual role of buoyancy forces in the context of the GSL is controversial (see e.g., [12–15]). Bekenstein [13] has recently shown that buoyancy protects the GSL, provided the floating point (see [14,12,13]) is close to the black-hole horizon. In addition, Bekenstein [13] has proved that one can derive the universal entropy bound Eq. (2) from the GSL when the floating point is near the horizon (this is the relevant physical situation for macroscopic and mesoscopic objects with a moderate number of species in the radiation, which seems to be the case in our world). The entropy bound Eq. (2) is also a *sufficient* condition for the validity of the GSL. For simplicity, and in the spirit of the original analysis of Bekenstein [11], we neglect buoyancy contribution to the energy bookkeeping of the body. As in the case of non rotating systems [13] we expect this not to effect the final entropy bound.

The gradual approach to the black hole must stop when the proper distance from the body’s center of mass to the black-hole horizon equals R , the body’s radius. Thus, in order to find the change in black-hole surface area caused by an assimilation of the spinning body, one should first solve Eq. (4) for E and then evaluate it at the point of capture $r = r_+ + \delta(R)$, where $\delta(R)$ is determined by

$$\int_{r_+}^{r_+ + \delta(R)} (g_{rr})^{1/2} dr = R , \quad (5)$$

with $g_{rr} = (r^2 + a^2 \cos^2 \theta) \Delta^{-1}$, and $\Delta = (r - r_-)(r - r_+)$. Integrating Eq. (5) one finds (for $\theta = \pi/2$ and $R \ll r_+$)

$$\delta(R) = (r_+ - r_-) \frac{R^2}{4r_+^2} . \quad (6)$$

Thus, the conserved energy E of a body having a radial turning point at $r = r_+ + \delta(R)$ [30] is

$$E = \frac{aJ}{\alpha} - \frac{Js(r_+ - r_-)r_+}{2\mu\alpha^2} + \frac{R(r_+ - r_-)}{2\alpha} \sqrt{\mu^2 + J^2 \frac{r_+^2}{\alpha^2}} , \quad (7)$$

where the “rationalized area” α is related to the black hole surface area A by $\alpha = A/4\pi$, and J is the body’s total angular momentum. The second term on the r.h.s. of Eq. (7) represents the above mentioned gravitational *spin-orbit* interaction between the orbital angular momentum of the body and its intrinsic angular momentum (spin).

An assimilation of the spinning body by the black hole results in a change $dM = E$ in the black-hole mass and a change $dL = J$ in its angular momentum. Using the first-law of black hole thermodynamics [6]

$$dM = \frac{\kappa}{8\pi} dA + \Omega dL , \quad (8)$$

where $\kappa = (r_+ - r_-)/2\alpha$ and $\Omega = a/\alpha$ are the surface gravity (2π times the Hawking temperature [31]) and rotational angular frequency of the black hole, respectively, we find

$$d\alpha = -\frac{2Jsr_+}{\mu\alpha} + 2R\sqrt{\mu^2 + J^2\frac{r_+^2}{\alpha^2}} . \quad (9)$$

The increase in black-hole surface area Eq. (9) can be *minimized* if the total angular momentum of the body is given by

$$J = J^* \equiv \frac{s\alpha}{Rr_+\sqrt{1 - \left(\frac{s}{\mu R}\right)^2}} . \quad (10)$$

For this value of J the area increase is

$$(\Delta A)_{min} = 8\pi\mu R\sqrt{1 - \left(\frac{s}{\mu R}\right)^2} , \quad (11)$$

which is the minimal increase in black-hole surface area caused by an assimilation of a spinning body with given parameters μ, s and R . Obviously, a minimum exists only for $s \leq \mu R$. Otherwise, ΔA can be made (arbitrarily) negative, violating the GSL. Moller's well-known theorem [32] therefore protects the GSL.

Arguing from the GSL, we derive an *upper bound* to the entropy S of an arbitrary system of proper energy E , intrinsic angular momentum s and proper radius R :

$$S \leq 2\pi\sqrt{(RE)^2 - s^2/\hbar} . \quad (12)$$

It is evident from this suggestive argument that in order for the GSL to be satisfied [$(\Delta S)_{tot} \equiv (\Delta S)_{bh} - S \geq 0$], the entropy S of the rotating system should be bounded as in Eq. (12). This upper bound is *universal* in the sense that it depends only on the *system's* parameters (it is *independent* of the black-hole parameters which were used to suggest it).

It is in order to emphasize an important assumption made in obtaining the upper bound Eq. (12); We have not taken into account *second-order* interactions between the particle's angular momentum and the black hole, which are expected to be of order $O(J^2/M^3)$. Taking cognizance of Eq. (10) we learn that this approximation is justified for rotating systems with negligible self gravity, i.e., rotating systems with $\mu \ll R$.

Although our derivation of the entropy bound is valid only for rotating systems with negligible self-gravity, we *conjecture* that it might be applicable also for strongly gravitating systems; A positive evidence for the validity of the bound is the fact that any Kerr black hole saturates it, provided the effective radius R is properly defined for the black hole: consider an electrically neutral Kerr black hole. Let its energy and angular momentum be $E = M$ and $s = Ma$, respectively. The black-hole entropy $S_{BH} = A/4\hbar = \pi(r_+^2 + a^2)/\hbar$ exactly saturates the entropy bound provided one identifies the *effective* radius R with $(r_+^2 + a^2)^{1/2}$, where $r_+ = M + (M^2 - a^2)^{1/2}$ is the radial Boyer-Lindquist coordinate for the Kerr black-hole horizon. The identification may be reasonable because $4\pi(r_+^2 + a^2)$ is exactly the black-hole surface area.

Evidently, systems with negligible self-gravity (the rotating system in our gedanken experiment) and systems with maximal gravitational effects (i.e., rotating black holes) both satisfy the upper bound Eq. (12). Thus, this bound appears to be of universal validity. The intriguing feature of our derivation is that it uses a law whose very meaning stems from gravitation (the GSL, or equivalently the area-entropy relation for black holes) to derive a universal bound which has *nothing* to do with gravitation [written out fully, the bound Eq. (12) would involve \hbar and c , but *not* G]. This provides a striking illustration of the *unity* of physics.

In summary, an application the generalized second law of thermodynamics to an idealized gedanken experiment in which an entropy-bearing rotating system falls into a black hole, enables us to conjecture an *improved upper bound* to the entropy of a *rotating* system. The bound is stronger than Bekenstein's bound for non-rotating systems. Moreover, this bound seems to be remarkable from a black-hole physics point of view: provided the effective radius

R is properly defined, *all* Kerr black holes *saturate* it (although we emphasize again that our specific derivation of the bound is consistent only for systems with *negligible* self gravity). This suggests that the Schwarzschild black hole is *not* unique from a black-hole entropy point of view, removing the disturbing feature of the entropy bound Eq. (2). Thus, *all* electrically neutral black holes seem to have the *maximum* entropy allowed by quantum theory and general relativity. This provides a striking illustration of the *extreme* character displayed by (all) black holes, which is, however, still *within* the boundaries of more mundane physics.

ACKNOWLEDGMENTS

I thank Jacob D. Bekenstein for helpful discussions. This research was supported by a grant from the Israel Science Foundation.

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